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Techniques to solve Non-homogeneous Quaternary Sextic Diophantine Equation

$$x^3 + y^3 = 7zw^5$$

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Abstract

New and fascinating patterns of integer solutions to non-homogeneous quaternary sextic diophantine equation given by $x^3 + y^3 = 7zw^5$ is obtained through employing substitution technique and factorization method.

Keywords: Quaternary sextic equation, Non-homogeneous sextic equation, Integer solutions, Substitution technique, Method of factorization

Introduction

It is well-known that a diophantine equation is an algebraic equation with integer coefficients involving two or more unknowns such that the only solutions focused are integer solutions .No doubt that diophantine equations are rich in variety [1-4] .There is no universal method available to know whether a diophantine equation has a solution or finding all solutions if it exists .For equations with more than three variables and degree at least three, very little is known. It seems that much work has not been done in solving higher degree diophantine equations. While focusing the attention on solving sextic Diophantine equations with variables at least three ,the problems illustrated in [5-25] are observed. This paper focuses on finding integer solutions to the non-homogeneous sextic equation with four unknowns $x^3 + y^3 = 7zw^5$ .In [25] ,a few sets of integer solutions to the above equation are obtained. In this paper, we present many more patterns of integer solutions for the above equation which are new and different from [25].

Method of analysis

The non-homogeneous polynomial equation of degree six with three unknowns to be solved for the integer solutions is

$$x^3 + y^3 = 7zw^5 \qquad (1)$$

The introduction of the transformations

$$x = u + v, y = u - v, z = 2u, u \neq v \quad (2)$$

in (1) leads to the sextic equation

$$u^2 + 3v^2 = 7 w^5 \quad (3)$$

The process of obtaining varieties of non-zero integer solutions to (1) is illustrated below:

Procedure 1

The substitution

$$u = kv, k > 1 \quad (4)$$

in (3) leads to

$$(k^2 + 3)v^2 = 7w^5$$

which is satisfied by

$$v = 7^3(k^2 + 3)^2 \beta^{5s} \quad (5)$$

and

$$w = 7(k^2 + 3)\beta^{2s}, \beta > 1, s > 0. \quad (6)$$

From (4) & (5) ,we get

$$u = 7^3 k(k^2 + 3)^2 \beta^{5s}$$

In view of (2) ,we have

$$\begin{aligned} x &= 7^3 (k+1)(k^2 + 3)^2 \beta^{5s}, \\ y &= 7^3 (k-1)(k^2 + 3)^2 \beta^{5s}, \\ z &= 2 * 7^3 k(k^2 + 3)^2 \beta^{5s}. \end{aligned} \quad (7)$$

Thus , (6) & (7) satisfy (1).

Procedure 2

The substitution

$$v = ku, k > 1 \quad (8)$$

in (3) leads to

$$(1+3k^2)u^2 = 7w^5$$

which is satisfied by

$$u = 7^3(1+3k^2)^2\beta^{5s} \tag{9}$$

and

$$w = 7(1+3k^2)\beta^{2s}, \beta > 1, s > 0. \tag{10}$$

From (8) & (9) ,we get

$$v = 7^3 k(1+3k^2)^2\beta^{5s}$$

In view of (2) ,we have

$$\begin{aligned} x &= 7^3(1+k)(1+3k^2)^2\beta^{5s}, \\ y &= 7^3(1-k)(1+3k^2)^2\beta^{5s}, \\ z &= 2*7^3\ (1+3k^2)^2\beta^{5s}. \end{aligned} \tag{11}$$

Thus , (10) & (11) satisfy (1).

Procedure 3

In (3) ,taking

$$v = w^2 \tag{12}$$

we get

$$u^2 = w^4(7w-3) \tag{13}$$

After some algebra , it is seen that the R.H.S. of (13) is a perfect square when

$$w = 7s^2 - 10s + 4 \tag{14}$$

and we have from (13) & (12)

$$u = (7s-5)(7s^2-10s+4)^2, v = (7s^2-10s+4)^2$$

In view of (2) , it is seen that

$$\begin{aligned} x &= (7s-4)(7s^2-10s+4)^2, \\ y &= (7s-6)(7s^2-10s+4)^2, \\ z &= 2(7s-5)(7s^2-10s+4)^2. \end{aligned} \tag{15}$$

Thus,(1) is satisfied by (14) & (15).

Note 1

In addition to (14) ,after some algebra , it is seen that the R.H.S. of (13) is a perfect square when

$$w = 7s^2 - 4s + 1 \tag{16}$$

and we have from (13) & (12)

$$u = (7s-2)(7s^2-4s+1)^2, v = (7s^2-4s+1)^2$$

In view of (2) , it is seen that

$$\begin{aligned} x &= (7s-1)(7s^2-4s+1)^2, \\ y &= (7s-3)(7s^2-4s+1)^2, \\ z &= 2(7s-2)(7s^2-4s+1)^2. \end{aligned} \tag{17}$$

Thus,(1) is satisfied by (16) & (17).

Procedure 4

The option

$$u = w^2 \tag{18}$$

in (3) gives

$$3v^2 = w^4(7w-1) \tag{19}$$

After some algebra , it is seen that the R.H.S. of (19) is a perfect square when

$$w = 21s^2 - 24s + 7 \tag{20}$$

and we have from (18) & (19)

$$u = (21s^2-24s+7)^2, v = (7s-4)(21s^2-24s+7)^2$$

In view of (2) , it is seen that

$$\begin{aligned} x &= (7s-3)(21s^2-24s+7)^2, \\ y &= (5-7s)(21s^2-24s+7)^2, \\ z &= 2\,(21s^2-24s+7)^2. \end{aligned} \tag{21}$$

Thus,(1) is satisfied by (21) & (20).

Note 2

In addition to (20) ,after some algebra , it is seen that the R.H.S. of (19) is a perfect square when

$$w = 21s^2 - 18s + 4 \tag{22}$$

and we have from (18) & (19)

$$u = (21s^2-18s+4)^2, v = (7s-3)(21s^2-18s+4)^2$$

In view of (2) , it is seen that

$$\begin{aligned} x &= (7s-2)(21s^2-18s+4)^2, \\ y &= (4-7s)(21s^2-18s+4)^2, \\ z &= 2\,(21s^2-18s+4)^2. \end{aligned} \tag{23}$$

Thus,(1) is satisfied by (22) & (23).

Procedure 5

Write (3) as

$$u^2 + 3v^2 = 7w^5 * 1 \tag{24}$$

Assume

$$w = a^2 + 3b^2 \tag{25}$$

Write the integers 7 & 1 in (24) as the product of complex conjugates as shown below:

$$\begin{aligned} 7 &= (2+i\sqrt{3})(2-i\sqrt{3}), \\ 1 &= \frac{(1+i\sqrt{3})(1-i\sqrt{3})}{4}. \end{aligned} \tag{26}$$

Substituting (25) & (26) in (24) and employing factorization , we consider

$$u+i\sqrt{3}\,v=(2+i\sqrt{3})(a+i\sqrt{3}\,b)^5\,\frac{(1+i\sqrt{3})}{2}$$
$$=\frac{(-1+i3\sqrt{3})}{2}[f(a,b)+i\sqrt{3}\,g(a,b)] \tag{27}$$

where

$$f(a,b)=a^5-30a^3b^2+45ab^4,$$
$$g(a,b)=5a^4b-30a^2b^3+9b^5.$$

Equating the coefficients of corresponding terms in (27) ,we have

$$u=\frac{1}{2}(-f(a,b)-9g(a,b)),\ v=\frac{1}{2}(3f(a,b)-g(a,b))$$

In view of (2) , we obtain

$$\begin{aligned} x&=f(a,b)-5g(a,b),\\ y&=-2f(a,b)-4g(a,b),\\ z&=-f(a,b)-9g(a,b). \end{aligned} \tag{28}$$

Thus , (1) is satisfied by (25) & (28).

Note 3

It is worth to mention that the integers 7 & 1 in (24) may also be expressed as the product of complex conjugates as shown below:

$$7=\frac{(5+i\sqrt{3})(5-i\sqrt{3})}{4},$$
$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4},$$
$$1=\frac{(1+i4\sqrt{3})(1-i4\sqrt{3})}{49},$$
$$1=\frac{(3r^2-s^2+i(2rs)\sqrt{3})(3r^2-s^2-i(2rs)\sqrt{3})}{(3r^2+s^2)^2},$$
$$1=\frac{(r^2-3s^2+i(2rs)\sqrt{3})(r^2-3s^2-i(2rs)\sqrt{3})}{(r^2+3s^2)^2}.$$

Following the procedure as above , some more sets of integer solutions to (1) are obtained by taking suitable combinations between 7 &1.

Conclusion:

In this paper ,an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous sextic diophantine equation with four unknowns given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the sextic diophantine equation with three or more unknowns.

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